PROPAGATION OF LONGITUDINAL AND TRANSVERSE SHOCK WAVES IN AN ELASTIC MEDIUM

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References [1-3] are devoted to the problem of the propagation of strong discontinuity surfaces in an elastic media. An incompressible medium is studied in [1], while [2, 3] consider an elastic body with a quasi Hooke's law relating stresses and the components of the Almansi strain. In the present paper we analyze the possibility of longitudinal shock waves in a medium with an elastic potential which can be written as a series in the principal invariants of the Almansi strain tensor. We study the effect of preliminary strain. The propagation of a pure transverse shock wave is shown to be possible for shear deformations in front of a strong discontinuity surface. The thermodynamic compatibility condition [2] is satisfied identically on the shock wave, and the velocity of propagation of a strong tangential discontinuity surface does not depend on the magnitude of the wave vector.

We assume that a strong discontinuity surface S moves with velocity G in a space filled with an elastic medium. We choose a fixed rectangular reference system x_1, x_2, x_3 with the x_3 axis along the direction of the outward normal to S. All equations will be written in this coordinate system.

1. We consider an elastic body with a potential W, which can be written in the form

$$W = \sum_{l,m,n} A_{lmn} J_1^l J_2^m J_3^n, \qquad A_{lmn} = \text{const} \tag{1.1}$$

$$J_1 = e_{kk}, \quad J_2 = {}^1/_2 (e_{kk}e_{jj} - e_{ij}e_{ij}), \quad J_3 = |e_{ij}|, \quad 2e_{ij} = u_{i,j} + u_{j,i} - u_{k,i}u_{k,j} \tag{1.2}$$

Here the u_i are the components of the displacement vector, the e_{ij} are the components of the Almani strain tensor, and the A_{lmn} are the elastic moduli.

The stresses and strains are related by [4]

$$\sigma_{ij} = \rho/\rho_0 \{(W_1 + W_2J_1 + W_3J_2 - 2W_3J_3)\delta_{ij} - (2W_1 + W_2 + 2J_1W_2 + W_3J_1)e_{ij} + (2W_2 + W_3)e_{ik}k_{kj}\} \tag{1.3}$$

$$W_k = \partial W / \partial J_k \quad (k = 1, 2, 3), \quad \rho = \rho_0 (1 - 2J_1 + 4J_2 - 8J_3)^{1/2}$$

where ρ and ρ_0 are, respectively, the densities of the body in the flowing and initial states, and W is the elastic potential.

Let S be a longitudinal shock wave. Then the geometric and kinematic compatibility conditions [5]

$$[u_{i,j}] = \omega_i \delta_{j3}, \qquad [\partial u_i / \partial t] = -G\omega_i \tag{1.4}$$

take the form

$$[u_{i,j}] = B\mu_{ij}, \quad [\partial u_i / \partial t] = -GB\delta_{i3}, \quad \mu_{ij} = \delta_{i3}\delta_{j3}, \quad B = \omega_3 \tag{1.5}$$

Here ω_i is the wave vector, $|B|$ is the strength of the longitudinal wave, and $[p] = p^- - p^+$, where p^- and p^+ are, respectively, the values of p behind and in front of S.

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Let

$$v_i = \partial u_i / \partial t + v_k \partial u_i / \partial x_k, \quad v_i^+ = 0, \quad u_{i,j}^+ = x\mu_{ij} \tag{1.6}$$

Then

$$e_{ij}^+ = (x - {}^1/_2 x^2)\, \mu_{ij} = c\mu_{ij} \tag{1.7}$$

On a shock wave Eq. (1.4) must be satisfied and, in addition conservation of mass and momentum [5] and a thermodynamic inequality [2] must hold

$$[\rho\theta] = 0, \quad [\rho\theta v_i] = -[\sigma_{i3}], \quad \theta = G - v_3 \tag{1.8}$$

$$\rho^+ (\theta^+)^2 [W'] \leqslant [\sigma_{i3}] ({}^1/_2 [\sigma_{i3}] + \sigma_{i3}^+) \tag{1.9}$$

where W' is the elastic potential per unit volume of the deformed body in front of S.

For a longitudinal shock wave Eqs. (1.8) and (1.9) simplify to

$$[\rho\theta] = 0, \quad \rho^+\theta^+ [v_3] = -[\sigma_{33}], \quad \rho^+ = \rho_0 (1 - x) \tag{1.10}$$

$$2(1 - x)(1 - x - B)[W] \leqslant B([\sigma_{33}] + 2\sigma_{33}^+) \tag{1.11}$$

Using (1.5) we find from (1.1), (1.2), (1.6) and (1.7)

$$\begin{gathered}
[e_{ij}] = (B - Bx - {}^1/_2 B^2)\, \mu_{ij} = a\mu_{ij}, \quad [\rho] = -\rho_0 B \\
[J_1] = a, \quad [J_2] = [J_3] = 0, \quad [J_1^k] = (a + c)^k - c^k = \varphi_k = a\psi_k \\
[W] = \sum_{n=2} \alpha_n \varphi_n, \quad [W_1] = \sum_{n=2} n\alpha_n \varphi_{n-1}, \quad [W_2] = \sum_{n=1} \beta_n \varphi_n \\
(1 - x - B)[v_3] = -GB, \quad \alpha_n = A_{n00}, \quad \beta_n = A_{n10}, \quad \alpha_0 = \alpha_1 = 0
\end{gathered} \tag{1.12}$$

Taking account of (1.12) we obtain from (1.3)

$$[\sigma_{33}] = B\left\{(1 - x - B)(1 - x - {}^1/_2 B) \sum_{n=2} n\alpha_n (\psi_{n-1} - 2\psi_n) - (1 - x)^2 \sum_{n=2} n\alpha_n c^{n-1}\right\} \tag{1.13}$$

Substituting (1.12) and (1.13) into (1.10) and (1.11) we find

$$\rho^+ G^2 = (1 - x - B) \sum_{n=2} n\alpha_n \{(1 - x - {}^1/_2 B)(1 - x - B)(\psi_{n-1} - 2\psi_n) - (1 - x)^2 c^{n-1}\} \tag{1.14}$$

$$\sum_{n=2} \alpha_n \{(1 - x - B)(1 - x + nB)\varphi_n - {}^1/_2 Bn (1 - x - B)\varphi_{n-1} - an(1 - x)^2 c^{n-1}\} \leqslant 0 \tag{1.15}$$

It follows from (1.14) that the velocity of a longitudinal shock wave propagating in an elastic body under conditions (1.6) is determined solely by the elastic moduli α_n. Only these coefficients enter inequality (1.15). The constants β_n also effect the values of $[\sigma_{11}]$ and $[\sigma_{22}]$. None of the remaining elastic moduli A_{ijk} from (1.1) affect the problem under consideration.

It follows from Eqs. (1.14) and (1.15), which must be satisfied on a shock wave of arbitrary strength, that for a certain fixed value of B there exists an $x = x_*$ such that for $x > x_*$ either inequality (1.15) is not satisfied or the square of the velocity of the shock wave G^2 in (1.14) becomes negative. Consider, for example, the propagation of weak longitudinal compression shock waves in an elastic body with a five-constant Murnaghan potential [4] ($\alpha_n = 0$ for $n \geq 4$, $|B| \ll x$), if $x \geq 0.1$. Since $\alpha_2 > 0$, when $\alpha_3 < 0$ we find $x_* = 0.225$; i.e., only for $x \leq 0.225$ can weak shock waves propagate in the body. If $\alpha_3 > 0$, $x_* = 0.225 + \mu$, where μ depends on the ratio α_3/α_2.

2. We consider the possibility of the propagation of a strong discontinuity surface S on which $[J_1] = [J_2] = [J_3] = 0$ in an elastic body with the potential (1.1). We note that certain forms of static deformations, which are possible in any incompressible elastic body when the invariants of the strain tensor are constants, are propounded in [6].

We assume that $u^+_{1,3} = z$, $u^+_{2,3} = y$, and all other $u^+_{ij} = 0$. Using (1.4) and (1.2) we obtain from the condition that J_1, J_2, and J_3 are constant in passing through S

$$\omega_3 = 0, \quad (\omega_1 + z)^2 + (\omega_2 + y)^2 = z^2 + y^2 \tag{2.1}$$

It follows from this that the end of the wave vector must lie in the (ω_1, ω_2) plane on a circle of radius $(z^2 + y^2)^{1/2}$ with its center at the point $(-z, -y)$. Taking account of (1.4) and (2.1) we find for velocity discontinuities

$$[v_1] = -G\omega_1, \; [v_2] = -G\omega_2, \quad [v_3] = 0 \tag{2.2}$$

and for stress discontinuities we have from (1.3)

$$[\sigma_{\alpha 3}] = Q\omega_\alpha \quad (\alpha = 1, 2), \quad [\sigma_{33}] = 0, \quad 4d = -(z^2 + y^2)$$
$$Q = -\left\{2 \sum_{\substack{l=1\\m=0}} lA_{lm0}(2d)^{l-1} d^m + \sum_{\substack{l=0\\m=1}} mA_{lm0}(2d)^l d^{m-1}\right\} \tag{2.3}$$

Taking account of (2.2) and (2.3) we obtain from the dynamic compatibility conditions (1.8)

$$\rho G^2 = Q \tag{2.4}$$

Since $[W] = 0$ and

$$[\sigma_{i3}]([\sigma_{i3}] + 2\sigma^+_{i3}) = Q^2(\omega_1^2 + \omega_2^2 + 2z\omega_1 + 2y\omega_2) = 0$$

the thermodynamic compatibility condition (1.9) is satisfied identically on the tangential strong discontinuity surface.

Thus the transverse wave is a degenerate shock wave since it, like a sound wave, is not accompanied by an entropy discontinuity. In addition it follows from (2.4) that the velocity of a transverse shock wave does not depend on its strength but is determined by the elastic moduli A_{lmn} and the values of the shear deformation in front of the wave. The strength of this wave can vary from 0 to $-8d$. For $z = y = 0$ the equality $\omega_1 = \omega_2 = 0$ must be satisfied, i.e., a transverse shock wave cannot propagate in an elastic body in the absence of a shear deformation in front of S.

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